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PROBLEM OF ADDITIONAL SOLUTIONS IN THE NONRELATIVISTIC AND RELATIVISTIC EQUATIONS

I.Statment of problem in the Scrhrodinger equation.

From the demand, that Hamiltonian and $p_r = -i\left(\frac{\partial}{\partial r} + \frac{1}{\partial r}\right)$ operators are Hermitian it follows, that [1.D.Blockincev, 2.V.Pauli, 3.A.Messia]

$$\lim_{r\to 0} rR = u(0) = 0 \qquad (1)$$

Usually are considered **regular** potentials in the Schrodinger equation

$$\lim_{r\to 0} r^2 V = 0 \qquad (2)$$

$$\underset{r\to 0}{R} = C_1 r^l + C_2 r^{-(l+1)} \qquad (3)$$

Second term in (3) doesn't obey (1) condition and is **neglected** usually **Singular** potentials

$$\lim_{r\to 0} r^2 V \to \pm\infty \tag{4}$$

Transition potentials

$$\lim_{r\to 0} r^2 V \to \pm V_0 \qquad (V_0 > 0) \tag{5}$$

Theorem. For transition potentials (with – sign in front of V_0) Schrodinger equation except standard solutions, may also have additional solutions.

Proof:
$$u'' + 2m[E - V(r)]u - \frac{l(l+1)}{r^2}u = 0; \quad u = Rr \tag{6}$$

At $r \to 0$ from (6) we obtain

$$u_{r\to 0} = a_{st} r^{\frac{1}{2}+P} + a_{add} r^{\frac{1}{2}-P} = u_{st} + u_{add} \tag{7}$$

Where

$$P = \sqrt{\left(l + \frac{1}{2}\right)^2 - 2mV_0} \tag{8}$$

In the region

$$\mathbf{0<P<1/2} \tag{9}$$

Both standard and additional solutions **satisfy** (!) condition (when **P>1/2** only **standard** solutions stay!)

From (8) and (9) we obtain condition of **exsitence** of additional states

$$l(l+1) < 2mV_0 \tag{10}$$

In [4-H.Bethe; R. Jackiw."Intermediate quantum mechanics"] is formulated very strong requirement- Kinetic Energy matrix elements should be finite! We show, that if we take "whole" wave function additional states sustain mentioned strong requirement!

Additional solutions satisfy also requirement, that [5.-L.Schiff.Quantum mechanics.] integral from particle coordinate probability density is finite!

Remark:

We think, that isn't correct paragraph 35- "Falling on the center"in [6-L.Landau, E.Lifchitz. Quantum mechanics].where is considered behavior of $R = \frac{u}{r}$ at small distances

$$\underset{r\to 0}{R} = Ar^{-\frac{1}{2}+P} + Br^{-\frac{1}{2}-P} \qquad (12)'$$

In (12)' both terms are singular (second term is more singular!) and in [7- R. Newton monograph] author notice: "If P<1/2, then the second solution is irregular in sense, that it is dominant above first solution". So R.Newton come very close to additional state problem, but don't mentioned that they exist! In [6] potential is made regular by cutting off it at some small r_0 and the limit $r_0 \to 0$ is taken, which selects less singular solution at $r_0 \to 0$ and so additional solutions are neglected! But if we multiple (12)' relation on r we get (7) relation, where we have, no singularity in the 0<P<1/2 region and as mentioned above u_{st} and u_{add} are "equal in rights" members of (7) relation!

I I. Introduction of self-adjoint extension τ parameter

It is well known, that for (4) and (5) type attractive potentials [8-K.Case.Phys.Rev.80,797(1950); 9-K.Meetz.Nuovo Cimento 34, 690(1964); 10-A.Perelomov, V.Popov.TMF.vol 4 (1970)] in the Schrodinger equation is shown, that it isn't enough to know potential and is necessary to introduce one arbitrary constant, which is equivalent to give boundary condition at the origin. Indeed, when

$$2mV_0 > (l+1/2)^2 \tag{11}$$

As one can see from (8) P is complex, both u_{st} and u_{add} solutions have same behavior at the origin and for example for $V = -\frac{g}{r^2}$ at small distances one have [7, 8]

$$u \approx A\sqrt{r}\cos\left(\sqrt{2mV_0 - (l+1/2)^2}\ln r + B\right) \tag{12}$$

Once B is arbitrary constant. On the Mathematical language it means, that H is symmetric (Hermitian), but isn't Self-adjoint operator and it is necessary to introduce 1 parameter for self-adjont extension(to make H Self-adjoint !)[11-M.Reed,B.Simon:vol 2]. As was shown in [8] if B is fixed constant, then all eigensolutions form a complete orthonormal set, and E-eigenvalues are real! (Once such a properties have a Self-adjoint H operator). But in this case we have "falling" on the center and energy isn't bounded from below!

In the region

$$2mV_0 < (l + 1/2)^2 \qquad (13)$$

based on the above mentioned paragraph of [7] , is neglected u_{add} solutions. We notice above, that u_{add} solutions in the $0<P<1/2$ region satisfy all u_{st} requirements. So is necessary to preserve it! Then for arbitrary E_1 and E_2 levels ortogonality condition is

$$m\left(E_2^2 - E_1^2\right)\int_0^\infty u_2 u_1 = 2P\left\{a_1^{st} a_2^{add} - a_2^{st} a_1^{add}\right\} \qquad (14)$$

And for ortogonality right side of (14) is zero

$$\frac{a_1^{st}}{a_1^{add}} = \frac{a_2^{st}}{a_2^{add}} \qquad (15)$$

So, we get, that for ortogonality it is necessary to introduce self-adjoint extension τ parameter

$$\tau = -\frac{a_{st}}{a_{add}} \tag{16}$$

All levels have same τ parameter. From (7) and (14) we have:

a) $a_{add} = 0$; $(\tau = -\infty)$ We keep only standard levels and they are orthogonal!

b) $a_{st} = 0$; $(\tau = 0)$ We keep only additional levels and they are orthogonal!

c) When $\tau \neq -\infty, 0$ then both levels exist at the same time!

For some unknown reasons the Nature choose only standard levels yet! We think, that other cases are also possible!

I I.I Model of Valent electron

$$V = -\frac{V_0}{r^2} - \frac{\alpha}{r}; \qquad V_0, \alpha > 0 \tag{17}$$

This potential "naturally" appears for coulomb potential in the Klein-Gordon equation. Following [12-W.Krolikowski; Bulletin De L' academics polonaise.Vol XVII.83(1979);13- A.A.Khelashvili,T.P.Nadareishvili, Bulletin of

Georgian Acad.Sci:Vol 164.no1(2001)] we obtain general solution of Schrodinger equation for (17) potential

$$u = C_1 \rho^{1/2+P} e^{-\rho/2} F\left(1/2 + P - \lambda, 1 + 2P; \rho\right) + C_2 \rho^{1/2-P} e^{-\rho/2} F(1/2 - P - \lambda, 1 - 2P; \rho) \quad (18)$$

Where P is given again by (8) and

$$\rho = \sqrt{-8mE} \cdot r; \qquad \lambda = \frac{2m\alpha}{\sqrt{-8mE}}; \quad E<0 \qquad (19)$$

From (18) wave function behavior at small r and (7) we obtain

$$\tau = -\frac{a_{st}}{a_{add}} = -\frac{C_1}{C_2}(-mE)^P \qquad (20)$$

(18) Wave function at large r should vanish and we get

$$C_1 \frac{\Gamma(1+2P)}{\Gamma(1/2+P-\lambda)} + C_2 \frac{\Gamma(1-2P)}{\Gamma(1/2-P-\lambda)} = 0 \qquad (21)$$

From (20) and (21) we get transcendental equation for E

$$\frac{\Gamma(1/2-\lambda-P)}{\Gamma(1/2-\lambda+P)} = \frac{1}{\tau}(-8mE)^P \frac{\Gamma(1-2P)}{\Gamma(1+2P)} \qquad (22)$$

E depends on τ parameter. In Two cases is possible to solve (22) analytically

a). $\tau = -\infty$. Then for standard levels determine condition is

$$1/2 - \lambda + P = -n_r \qquad n_r = 0,1,2,... \tag{23}$$

b) $\tau = 0$. Then for additional levels determine condition is

$$1/2 - \lambda - P = -n_r \qquad n_r = 0,1,2,... \tag{24}$$

So in these two cases we have

$$E_{st,add} = -\frac{m\alpha^2}{2[1/2 + n_r \pm P]^2} = -\frac{m\alpha^2}{2\left[1/2 + n_r \pm \sqrt{(l+1/2)^2 - 2mV_0}\right]} \tag{25}$$

Remark: For $V_0 < 0$ in (17) , we get Kratzer Molecular potential and we obtain for standard levels well known formula, but in this case isn't fulfilled (10) condition and so we have no additional levels for Kratzer potential .

For alkaline metal atoms (Li,Na,K,Rb,Cs) is used (17) potential [14-S.Frish . Optical specra of atoms;15 –M.Eliashevich.Atomic and molecular spectroscopy].Spectra of this atoms is similar hydrogen atom spectra

$$E_{n'} = -R\frac{1}{n'^2} \tag{26}$$

Where R is Rydberg constant and n' is effective principal number

$$n' = n_r + l' + 1 \quad (27)$$

And l' is defined from

$$l'(l'+1) = l(l+1) - 8mV_0 \quad (28)$$

For l' is taken only + sign in front of root (P)[14,15]

$$l' = -1/2 + P = -1/2 + \sqrt{(l+1/2)^2 - 2mV_0} \quad (29)$$

(26) Is just (25) for E_{st}. So up to now wasn't considered additional levels (- sign in front of root). Then in [14] the root is expand is expand for small V_0

$$E_{st} = -R\frac{1}{\left(n+\Delta_l^{st}\right)^2}; \qquad n = n_r + l + 1 \quad (30)$$

Where Δ_l^{st} is Rydberg correction (quantum defect)

$$\Delta_l^{st} = -\frac{2mV_0}{2l+1} \quad (31)$$

For E_{add} we can't take small V_0 ,because $l(l+1) < 2mV_0$. So for E_{st} at $V_0 \to 0$ one get hydrogen atom spectra; E_{add} exist only for "strong" values of V_0 !

Now we can rewrite (25) formula

$$E_{st,add} = R_0 \frac{1}{(n - 1/2 \pm p - l)^2}; R_0 = R; \; n = n_r + l + 1 \quad (32)$$

It is clear, that $E_{st} > E_{add}$ and when n increase, E_{add} approach to E_{st} from below.

We can write (32) in (30) form

$$E_{st} = R_0 \frac{1}{\left(n + \Delta_l^{st}\right)^2} \quad (33)$$

Where $$\Delta_l^{st} = P - (l + 1/2) \quad (34)$$

From (34) and (8) P definition we get E_{add} existence condition

$$-(l+1) < \Delta_l^{st} < -l \quad (35)$$

$$E_{add} = \frac{R_0}{\left(2n - 2l - 1 - \sqrt{\frac{R_0}{E_{st}}}\right)} \quad (36)$$

We see that if one knows E_{st} levels, we can find also E_{add} and we calculate for some alkaline metal atoms these levels. So it is expectable, that in the Model of Valent electron, beside the well known E_{st} levels, may also exist E_{add} and (22) transcendental equation levels (It depends on τ self-adjoint parameter value).

Remark: Our formalism works everywhere, where (17) potentials works: for excited (Rydberg) atoms, for alkaline isoelectronic ions and etc.

I V . Singular (Spiked) Oscillator model

$$V = -\frac{V_0}{r^2} + gr^2; \quad V_0, g > 0 \quad (37)$$

Use: Calogero model, Fractional statistics and anyons, Quantum Hall effect, Spin chains, Two dimensional QCD.

General solution is

$$U = e^{-\frac{\sqrt{2mg}}{2}r^2}\left\{C(2mg)^{\frac{-1/2+P}{4}}r^{1/2+P}F\left(-n,1+P;\sqrt{2mg}r^2\right)+D(2mg)^{\frac{-1/2-P}{4}}r^{1/2-P}F\left(-n-P,1-P;\sqrt{2mg}r^2\right)\right\}$$

Where $\sqrt{\frac{2m}{g}}E = 4(n+s) = 3 \qquad s = \frac{1}{2}\left(-\frac{1}{2}+P\right)$ (38)

By using the same method as in model of valent electron, we obtain

$$\frac{\Gamma\left(-\frac{1}{4}\sqrt{\frac{2m}{g}}E+\frac{1}{2}-\frac{P}{2}\right)}{\Gamma\left(-\frac{1}{4}\sqrt{\frac{2m}{g}}E+\frac{1}{2}+\frac{P}{2}\right)} = \frac{1}{\tau}(2mg)^{P/2}\frac{\Gamma(1-P)}{\Gamma(1+P)} \qquad (39)$$

Where now $\tau = -\frac{a_{st}}{a_{add}} = -\frac{C}{D}(-2mg)^{P/2}$ (40)

For $\tau = -\infty$ and $\tau = 0$ we get standard and additional levels

$$E_{st,add} = 2\sqrt{\frac{g}{2m}}\{2n_r + 1 \pm P\} \qquad n_r = 0,1,2... \qquad (41)$$

We can write (41) so

$$E_{st,add} = K\left\{n + 3/2 \pm P - \left(l + 1/2\right)\right\} \tag{42}$$

Where $K = 2\sqrt{\frac{g}{2m}}$ and $n = 2n_r + l.$

Again quantum defect is defined by (34), for E_{add} existence one has (35) condition and

$$E_{add} = K\left(2n + 2 - 2l - \frac{E_{st}}{K}\right) \qquad E_{st} > E_{add}$$

Remarks: 1. for $V = -\frac{V_0}{r^2} + W(r)$ potential (where W is regular potential) we can define generally Δ_l^{st} quantum defect by $\Delta_l^{st} = P - (l + 1/2)$ as a deviation from W(r), because when $V_0 = 0$, then P=l+1/2 and Δ_l^{st} =0.

2. !n [16- K.Gupta;B.Basu-mallick; Phys.lett B 526,121(2002); Phys.Lett A; V311,87 (2003); Phys.Lett A323,29(2004)] is considered rational Calogero model for N particles

$$\hat{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \left[\frac{a^2 - 1/4}{(x_i - x_j)^2} + \frac{\Omega^2}{16} (x_i - x_j)^2 \right] \tag{43}$$

a, Ω Are constants, x_i coordinate of i-particle. $\hat{H}$ is a Hermitian operator. To determine whether $\hat{H}$ is self-adjoint , we have to look for square integrable of the equations

$$\widetilde{H}^* \Phi_{\pm} = \pm i \Phi_{\pm} \tag{44}$$

Where $\hat{H}^*$ is the adjoint of $\hat{H}$. The domain $D_z(\widetilde{H})$ in which $\hat{H}$ is self-adjoint contains all the elements of $D(\widetilde{H})$ together with elements of the form $\Phi_+ + e^{iz}\Phi_-$, where z is self-adjoint extension parameter. Eigenvalue equation

$$\frac{\Gamma\left(\frac{1-\nu}{2} - \frac{E}{4\omega}\right)}{\Gamma\left(\frac{1+\nu}{2} - \frac{E}{4\omega}\right)} = \frac{\xi_2 \cos\left(\frac{z}{2} - \eta_1\right)}{\xi_1 \cos\left(\frac{z}{2} - \eta_2\right)} \tag{45}$$

a). $z = z_1 = \pi + 2\eta_1$ b). $z = z_2 = \pi + 2\eta_2$ (46)

$E_n = 2\omega(2n + \nu + 1)$ $E_n = 2\omega(2n - \nu + 1)$

For N=2 we have such formal correspondence between (37) potential and Calogero Model

$\nu \to P; \omega^2 / 4 \to g$ and $-\frac{1}{4}\sqrt{\frac{2m}{g}}E + \frac{1}{2} - \frac{P}{2} \to \frac{1-\nu}{2} - \frac{E}{2\omega}$ .So (39) and (45) equations left sides are almost identical., but different are right sides .It was expectable, because we consider 3-dimensional case and the Calogero model is one dimensional. It is interesting, that (46) is similar our E_{st}, E_{add} .It should be mentioned, that (45) equation is obtained by general mathematical theory of Self-adjoint operators [11] and our (39) by alternative quick and simple procedure, leading to the same results –so called "Pragmatic approach" [17-J.Audretsh; J.Phys. A34,235(2001)].The point is that we demand ortogonality of different states and by self –adjoint extensions of operators is reached ortogonality. It should be mentioned also, that in (39) and (45) equations for $\tau \neq 0, -\infty$ and $z \neq z_1, z_2$ energy levels have nonequispaced nature!

3.In N dimensional case one have

$$P = \sqrt{\left[l + \frac{1}{2}(N-2)\right]^2 - 2mV_0} \tag{47}$$

$$\left[l + \frac{1}{2}(N-2)\right]^2 - \frac{1}{4} < 2mV_0 \tag{48}$$

As we see from (48) with increasing of N, is increasing restrictions on V_0 from below. For example when $l = 0$

$$\frac{(N-1)(N-3)}{4} < 2mV_0$$

When N>3, for $l = 0$, V_0 isn't small (For N=3 it is possible). High dimensions are considered in many body problems in so called "Hypersperical formalism" and also now is very popular extra dimensions problems.

V . Modification of Van-Roen-Weiscof Formula.

Decay widths are $\Gamma \sim |R_s(0)|^2$, for $l = 0$ states, $\Gamma \sim |R_p^{/}(0)|^2$ for $l = 1$, $\Gamma \sim |R_D^{//}(0)|^2$ for $l = 2$ states and etc when V is regular. But for $\lim_{r\to 0} r^2 V \to -V_0$ potentials it is shown [7], that $|R_s(0)|$ is divergent. So it is necessary that to modify Van-Roen-Waiscof formula. We solve this problem.

a).Hypervirial theorem

$$u'' + L(r)u = 0 \tag{49}$$

$$\lim_{r\to\infty} f[u']^2 \to 0; \qquad \lim_{r\to\infty} fLu^2 \to 0 \tag{50}$$

$$\{fu'^2 - f'uu' + 1/2 f''u^2 - fuu''\}_{r=0} = -2\langle f'L\rangle - \langle fL'\rangle - 1/2\langle f'''\rangle \tag{51}$$

This is Generalized Virial or Hypervirial Theorem. In the literature is considered only $f = r^q (q \geq -2l)$ and Schrodinger equation case, when

$$L = 2m\left[E - V - \frac{l(l+1)}{2mr^2}\right] \tag{52}$$

[18-C.Quigg;J.Rosner.Physics Reports 56,167(1979);19-H.Grosse.,A.Martin. Physics Reports 60,341(1980);] So (51) is most general and powerful relation!

b). Wave function at the origin

$$L = A(r) - \frac{l(l+1)}{r^2} \quad l = 0,1,2... \tag{53}$$

1). $\lim_{r\to 0} r^2 A(r) = 0$ (54)

$$\lim_{r\to 0} u_l = a_l r^{l+1} \tag{55}$$

From (51) we have

$$a_l^2 \left\{r^{2l}\left[(l+1)f - (l+1)f'r + r^2/2f''\right]\right\}_{r=o} = -2\langle f'A\rangle - \langle fA'\rangle + 2l(l+1)\left\langle \frac{f'}{r^2} - \frac{f}{r^3}\right\rangle - 1/2\langle f'''\rangle \tag{56}$$

For $f = r^{-q}$ we obtain from (56)

$$(2l+1)^2 a_l^2 \delta_{q,-2l} = -\langle 2qr^{q-1}A + r^q A'\rangle + \left[2l(l+1)(1-q) + 1/2q(q-1)(q-2)r^{q-3}\right] \tag{57}$$

When $q = -2l$, noticing that $\psi(\vec{r}) = R_{nl}(r)\mathrm{Y}_{lm}(\theta,\varphi) = \frac{u_{n,l}(r)}{r}\mathrm{Y}_{lm}(\theta,\varphi)$, we obtain

$$(2l+1)^2 \left| R_{nl}^{(l)}(0) \right|^2 = (l!)^2 \left\langle 4l \frac{A}{r^{2l+1}} - \frac{A'}{r^{2l}} \right\rangle_{nl} \quad (58)$$

Remark: (58) is a generalization of [20-A.Khare.Nuclear Physics B152(1979)] article formula, where is considered only Schrodinger equation A=2m(E-V).

2). $\lim_{r \to 0} r^2 A(r) = -V_0; \quad V_0 > 0$ (59)

$$u_{r \to 0} = a_{st} r^{\frac{1}{2}+P} + a_{add} r^{\frac{1}{2}-P}; \; 0 < P < 1/2 \quad (60)$$

In this case is very difficult to obtain something for whole (60) function. and we consider only $u_{st}(\tau = -\infty)$ and $u_{add}(\tau = 0)$ cases.

$$u_l^{st,add} \underset{r \to 0}{=} a_l^{st,add} r^{\frac{1}{2} \pm P} \quad (61)$$

We take now $f = r^{1\mp 2P}$ - for standard and + for additional states.

$$4P^2 a_l^2 = -2(1\pm 2P)\left\langle r^{\pm 2P} A\right\rangle - \left\langle r^{1\pm 2P} A'\right\rangle + \left[2(1\pm 2P)l(l+1) - 2l(l+1) \pm (1+2P)P(1-2P)\right]\left\langle r^{\pm 2P-2}\right\rangle \quad (62)$$

c).Modification of Van-Roen-Weiscoff formula $(l = 0)$

From (61) $$a_0^{st} = \left[u_{st} r^{-\frac{1}{2}-P}\right]_{r=0} = \left[R_{st} r^{\frac{1}{2}-P}\right]_{r=0} \quad (63)$$

For Schrodinger equation $P = \sqrt{1/4 - 2mV_0} < 1/2$. In (63) is infinite, but $r^{\frac{1}{2}-P}$ is finite (P<1/2) and a_0^{st} is finite! For $V_0 = 0$ (Case (54)) in Weiscoff formula $\left|R_0^{st}(0)\right|^2$ is considered, which is (57) relations left side for $l = q = 0$. We assume:

For (59) case in Weiscoff formula we take regularized $\left|R_{0,st}^{reg}(0)\right|^2$ expression [(62) relation left side!]

$$\left|R_{0,st}^{reg}(0)\right|^2 = 4P^2\left[R_{0,st}(r)r^{\frac{1}{2}-P}\right]_{r=0}^2 \qquad (64)$$

Remarks: 1. For $l \neq 0$ we have more complicated calculations and we get

$$\left|R_{l,st}^{(l),reg}(0)\right|^2 = \left|R_{l,st}^{(l)}r^{\frac{1}{2}+l-P}\right|_{r=0}^2 \qquad (65)$$

Where $R^{(l)}$ denotes l order derivation.

2. For $V_0 = 0, P = l + 1/2$ and from (65) $\left|R_{l,st}^{(l),reg}(0)\right|^2 = R_{l,st}^{(l)}(0)$ is finite!

3. For additional states $P \to -P$ change should be done in (64) and (65).

VI. Two- body Klein-Gordon equation

$$\left\{\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{V^2}{4} - \frac{MV}{2} + \frac{M^2}{4} - m^2\right\}u = 0 \quad \text{M=2m+E –total mass} \qquad (66)$$

In this case Transition potentials is

$$\lim_{r \to 0} r\ V \to \pm V_0 \quad (V_0 > 0) \tag{67}$$

And one has additional levels. For example for $V = -\frac{V_0}{r}$ we obtain [13]

$$M_{st,add} = \frac{2m}{\sqrt{1 + \frac{(V_0 / \ 2)^2}{(n_r + 1/2 \pm P)^2}}} \; ; \; P = \sqrt{(l + 1/2)^2 - V_0^2 / 4} \tag{68}$$

For (π^+, π^-) systems Coulomb interaction have some meaning and M_{add} can be founded in experiments! (We find also M_{add} for Hulten potentials.). M_{add} States existence condition is

$$4l(l+1) < V_0^2 < 4l(l+1) + 1 \tag{69}$$

1). Nonrelativistic limit.

a). $l = 0; n_r \neq 0$ $\qquad 0 < V_0 < 1$

$$M_{st} = 2m - \frac{V_0^2 m/2}{2(n_r+1)^2} + \frac{3}{4}\cdot\frac{V_0^4 m/2}{8(n_r+1)^4} + O(V_0^6) \qquad (70)$$

$$M_{add} = 2m - \frac{V_0^2 m/2}{2n_r^2} + \frac{3}{4}\cdot\frac{m}{2}\cdot\frac{V_0^4}{n_r^2}\left\{\frac{3}{128} + \frac{1}{n_r}\right\} + O(V_0^6) \qquad (71)$$

In V_0^2 order we have Balmer formula for standard levels and $M_{1.st} = M_{2,add}; M_{2,st} = M_{3,add}$, so it is impossible to distinguish standard and additional levels in V_0^2 order (it is possible only by n_r nodes!) and in V_0^4 order distinction is obvious.

b). $l = n_r = 0$

$$M_{st,add} = \sqrt{2}m\sqrt{1 \pm \sqrt{1 - V_0^2}} \qquad (72)$$

$$M_{st} = 2m - \frac{m}{2}\cdot\frac{V_0^2}{2} - \frac{m}{2}\cdot\frac{5\cdot V_0^4}{32} + O(V_0^6) \qquad (73)$$

$$M_{add} = mV_0 + \frac{mV_0^3}{8} + O(V_0^5) \tag{74}$$

c). $l \neq 0$

For M_{st} we have no restriction from below, while for M_{add} is restricted by (69) relation from below. So we can expand only standard levels! Physically it means, that additional levels may appear only in "strong" fields, this means, that this case is relativistic and isn't possible nonrelativistic consideration!

VII. Problem of additional solutions for high spins

1). One body Dirac equation.

$$G' + \chi / r \cdot G - (E + m - V)F = 0 \tag{75}$$

$$F' - \chi / r \cdot F + (E - m - V)G = 0 \tag{76}$$

$$G = \sqrt{E + m - V} \cdot \varphi \tag{77}$$

$$\varphi'' + \left\{ (E-V)^2 - m^2 - \frac{\chi(\chi+1)}{r^2} \right\} \varphi = \left\{ \frac{3}{4} \frac{V'^2}{(E-V+m)^2} + \frac{V'' - \frac{2\chi}{r} V'}{2(E-V+m)} \right\} \varphi \quad 78)$$

At small r we get

$$\varphi'' + \frac{V_0^2 - \chi^2 + 1/4}{r^2} \varphi = 0 \qquad (79)$$

$$\varphi \sim r^{\frac{1}{2} \pm P}; P = \sqrt{\chi^2 - V_0^2} \qquad (80)$$

In (80) relation is possible P<1/2 and as if one have additional solutions, but from (77) we see

$$\underset{r \to 0}{G} \sim \sqrt{-V} \cdot \varphi = \frac{\sqrt{V_0}}{\sqrt{r}} r^{\frac{1}{2} - P} = r^{-P} \qquad (81)$$

G is divergent (isn't fulfilled $G \to 0$ fundamental condition) and so we have no additional levels!

2).Breit equation

$$\left\{\frac{d^2}{dr^2}-\frac{J(J+1)}{r^2}+\frac{V'}{M-V}\cdot\left(\frac{d}{dr}-\frac{1}{r}\right)+\frac{(M-V)^2-4m^2}{4}\right\}F=0 \tag{82}$$

There are no additional levels!

3).Proca equation

One have two cases [8, 21-V.S.Popov.Nucl.Phys.vol14.(1973)]

a). $l=j$, or $l=0, j=1$. In this case Proca equation is reduced to Klein-Gordon equation and so for (67) transition potentials additional levels exist!

b). $l=j\pm 1, j\geq 1$ .at small r one have [8,21]

$$u_{1,2}^{//}+f_{1,2}(r)u_{1,2}=0 \tag{83}$$

$$f_{1,2}=\mp\frac{\sqrt{J(J+1)}}{r}V'(r)-\frac{J(J+1)}{r^2} \tag{84}$$

Here u_1 corresponds $l=J+1$ and u_2 to $l=J-1$. From (84) is clear, that for $V=gr^n$ potentials for $n\neq 0$ one have “falling” onto center! In this case transition potential is logarithmic potential (n=0) $V=V_0\ln r$ and for $L=J+1$ $P=\sqrt{(J+1/2)^2+\sqrt{J(J+1)}V_0}>1/2$ and we have no additional states and for

$L = J - 1 \quad P = \sqrt{(J+1/2)^2 - \sqrt{J(J+1)}V_0} < 1/2$ and for $\sqrt{J(J+1)} < V_0$ additional states exist!

VIII. Concluding remarks. Summary

1.Divergence of full wave function at origin $\psi(0)$ is necessary condition of existence of additional states, but not sufficient! Indeed for Schrodinger,Klein-Gordon equations we have additional states and $\psi(0)$ is infinite, but for Dirac and Breit equations $\psi(0)$ are also infinite, but one have no additional states or in other words we can say: if additional states exist, then without fail $\psi(0)$ is divergent for standard and additional states!

2. It is necessary to investigate more carefully dependence of additional states on space dimension. In [22-B.Basu-mallick Phys. Rev.B62,99927; Int.J.Mod. Phys.B16.1875 (2002)] is noticed ,that in one dimensional Calogero model ν parameter is given by

$$\nu = \frac{1}{2}\left[1 \pm \sqrt{1+4g}\right]$$

Here new moment is, that – sign is taken in front of root (as we take –sign in 3-dimensional case!).in original article [23-F.Calogero.J.math.Phys.2191(1969)] –sign is neglected!

In one dimensional Dirac equation, we think additional states exist and this question is considered in [24-A.S.De Castro.Annalys.Phys.311,170 (2004)],where author don't say directly that additional states exist! Now we have no $\frac{2}{r}G'$ term in Dirac equation and it isn't necessary (77) transformation.

3. We think, that additional solutions exist also in no stationary problems [25-V.Dodonov.phys.Rev A57,2851 (1997), B.Samsonov.quant-.ph/0401093]

$$V = -g / r^2 + k(t) r^2$$

4.Our main result: We show, that for $\lim_{r\to 0} r^2 V \to -V_0 \ ; (V_0 > 0)$ potentials in the region $(l+1/2)^2 > 2mV_0$ (no "falling onto center!) it is necessary to keep second additional solution in the 0<P<1/2 interval (We have our variant of

Landay mentioned paragraph!) and it is also necessary to introduce self-adjoint extension τ parameter $\tau = -\frac{a_{st}}{a_{add}}$.

We have two possibilities

1).It should be found another strong requirement in the quantum mechanic mathematical formalism, which "destroys" additional states!

2) We should admit, that in the region $(l+1/2)^2 > 2mV_0$ H isn't self-adjoint,it is necessary it extension by introducing τ parameter. Finally we can say, that well known problem in the "opposite" region $(l+1/2)^2 < 2mV_0$, take place also in our region and it stay open the following questions: Why the NATURE "select" only standard states $(\tau = -\infty)$?!Is it possible to discover additional solutions in future experiments?!